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# A symmetric characteristic FVE method with second order accuracy for nonlinear convection diffusion problems ☆

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**Abstract**

The finite volume element (FVE) methods used currently are essentially low order and unsymmetric. In this paper, by biquadratic elements and multistep methods, we construct a second order FVE scheme for nonlinear convection diffusion problem on nonuniform rectangular meshes. To overcome the numerical oscillation, we discretize the problem along its characteristic direction. The choice of alternating direction strategy is critical in this paper, which guarantees the high efficiency and symmetry of the discrete scheme. Optimal order error estimates in $H^1$-norm are derived and a numerical example is given at the end to confirm the usefulness of the method.


*MSC:* 65N30; 65N12

*Keywords:* Finite volume element; Second order; Symmetric scheme; Error estimates

## 1. Introduction

Finite volume element method [5,18,11,7], also named as generalized difference method [16,15] or box method [1], has a long history as a discretization tools for the numerical simulation of various conservation laws. The method involves two spaces: the solution space of piecewise polynomial functions over the primal partition, and the test space of piecewise constant functions over the dual partition. Similar as the finite element method the unknowns are approximated by a Galerkin expansion. The popularity of this method stems from the structural simplicity and the presence of local conservation properties of the numerical fluxes.

But some undesirable features, such as low order spatial convergence and nonsymmetric of the discrete scheme, limit the application of FVE method. One approach to obtain high order schemes is using uniform or symmetric meshes to obtain superconvergence. The FVE approximations for elliptic and integro differential equations on such meshes with second-order spatial convergence rate have been considered in [18,11]. Another approach is choosing high

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order finite elements as solution space. FVE discretization for Poisson equations, with quadratic finite elements on triangulations, was first proposed in [21]. Then the method was extended to analyze more complicated cases in [6,22]. However, the schemes are not symmetric even for constant coefficient problems. The literature on the symmetric FVE methods is little. By the lumped mass methods and approximation replacement of the weak forms, some symmetric schemes for linear elliptic and parabolic problems were constructed in [14,17,19], where the spatial convergence rates are first order.

Thus, the main goal of this paper is to develop a high order symmetric FVE method to handle general nonstationary problems. We choose the biquadratic elements as the solution space and put forward a new dual partition to guarantee the symmetry of the both corresponding bilinear form and inner product. In order to avoid the nonphysical oscillation and numerical dispersion, an efficient approximate procedure–characteristic based method [20,8,12] is adopted. Since in some cases, the convection diffusion is sensitive to time, it is necessary to consider more accurate full discretization in time. So a multistep method [2,3] is used in this paper to raise the accuracy along the temporal direction to the second order.

Let us consider the following nonlinear convection diffusion problem:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \nabla \cdot (a(X,u)\nabla u) + b(X,u) \cdot \nabla u = f(X,t,u), \quad (X,t) \in \Omega \times (0,T],\\ &u = 0, \quad (X,t) \in \partial\Omega \times (0,T], \\ &u(X,0) = u_0(X), \quad X \in \Omega,\end{aligned} \tag{1.1}$$

where $\Omega = (a_x, b_x) \times (a_y, b_y)$ with its boundary $\partial\Omega$ and $X = (x, y)$. Next, we make some assumptions about the data.

**Assumption 1.** For any $(X, u) \in \Omega \times R$,

1. $0 < a_* \leqslant a(X,u) \leqslant a^*$, $a(X,u)$ is $\varepsilon_0$-Lipschitz continuous with respect to $X$, and $|(\partial^i/\partial u^i)a(X,u)| \leqslant a^*$, $i = 1, 2$;
2. $|b(X,u)| \leqslant b^*$, $b(X,u)|_{\partial\Omega} = (0,0)$, and $b(X,u)$ is $\varepsilon_0$-Lipschitz continuous with respect to $X$, $u$;
3. $f(X,t,u)$ is $\varepsilon_0$-Lipschitz continuous with respect to $X$, $u$,

where $a_*, a^*, b^*$ are positive constants. We say that a given function $f(X,u)$ is $\varepsilon_0$-Lipschitz continuous with respect to $X$ and $u$, provided there exists a constant $L > 0$ such that $|f(X,u) - f(Y,v)| \leqslant L(|X-Y| + |u-v|)$ if $|X-Y| \leqslant \varepsilon_0$ and $|u-v| \leqslant \varepsilon_0$.

**Remark 1.** We assume that $b(X,u)|_{\partial\Omega} = (0,0)$ only to simplify the error estimates. For general $b$, we can consider the periodic boundary condition or make the spatial interpolation on the boundary [20,8,12].

Let $\psi(X,u) = [1 + |b(X,u)|^2]^{1/2}$. Denote by $\tau$ the characteristic direction of $(\partial u/\partial t) + b(X,u) \cdot \nabla u$. Then

$$\psi(X,u)\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t} + b(X,u) \cdot \nabla u.$$

Now we can write the first equation of (1.1) in its characteristic form

$$\psi(X,u)\frac{\partial u}{\partial \tau} - \nabla \cdot (a(X,u)\nabla u) = f(X,t,u), \quad (X,t) \in \Omega \times (0,T]. \tag{1.2}$$

Throughout this paper, unless stated otherwise, $M$ and $\varepsilon$ will denote a general positive constant and general positive small constant, respectively, not necessarily the same in different places.

## 2. The meshes and notation

In this section, we will use the label $\lambda$ to denote the spatial coordinates $x$ or $y$ for brevity.

First, we introduce a nonuniform primal mesh $T_h$ of the domain $\Omega$ as the tensor product $T_{x,h} \times T_{y,h}$. To define $T_{\lambda,h}$ ($\lambda = x, y$), we choose $2n_\lambda + 1$ grid points distributed in $\lambda$ directions as follows:

$$T_{\lambda,h}: a_\lambda = \lambda_0 < \lambda_{1/2} < \lambda_1 < \lambda_{3/2} < \cdots < \lambda_{n_\lambda-(1/2)} < \lambda_{n_\lambda} = b_\lambda,$$

where $\lambda_{i+(1/2)} = \frac{1}{2}(\lambda_i + \lambda_{i+1})$ $(0 \leqslant i < n_\lambda)$.

Next, we introduce a dual mesh $T_h^* = T_{x,h}^* \times T_{y,h}^*$ based on $T_h$. Let

$$T_{\lambda,h}^*: a_\lambda = \lambda_0 < \lambda_{1/6} < \lambda_{5/6} < \cdots < \lambda_{n_\lambda - \frac{1}{6}} < \lambda_{n_\lambda} = b_\lambda,$$

where $\lambda_{i+(1/6)} = \lambda_i + \frac{1}{6} h_{\lambda,i+1}$ $(0 \leqslant i < n_\lambda)$ and $\lambda_{i+(5/6)} = \lambda_i + \frac{5}{6} h_{\lambda,i+1}$ $(0 \leqslant i < n_\lambda)$. Here

$$h_{\lambda,i+1} = \lambda_{i+1} - \lambda_i.$$

**Remark 2.** Generally, the dual mesh is made by the midpoints of all neighboring primal nodes [16,5]. The aim of our new formulation is to ensure the symmetry of the inner product $(u_h, \Pi^* v_h)$ and the bilinear form $A(u_h, \Pi^* v_h)$, which is to be defined later.

In this paper, we consider the regular partition, i.e., there exists a positive constant $\mu$ such that

$$\mu h \leqslant h_{x,i},\ h_{y,j} \leqslant h, \quad i = 1, \ldots, n_x, \quad j = 1, \ldots, n_y, \tag{2.1}$$

where $h = \max\{h_{x,i}, h_{y,j}\}$.

Let $N_h$ denote the set of all nodes of $T_h$. For any node $P = (x_{i/2}, y_{j/2}) \in N_h$ $(0 \leqslant i \leqslant 2n_x,\ 0 \leqslant j \leqslant 2n_y)$, we denote by $K_P^*$ the control volume of node $P$ such that

$$K_P^* = [x_{(i/2)-(1/4)+((-1)^i/12)},\ x_{(i/2)+(1/4)-((-1)^i/12)}] \times [y_{(j/2)-(1/4)+((-1)^j/12)},\ y_{(j/2)+(1/4)-((-1)^j/12)}].$$

Here the control volume should be modified correspondingly for the boundary nodes. For instance, the control volume of $(x_0, y_0)$ is $[x_0,\ x_{1/6}] \times [y_0,\ y_{(1/6)}]$.

Now from (1.1), (1.2) and the Green's formula, we obtain

$$\int_{K_p^*} \psi(X,u)\frac{\partial u}{\partial \tau}\,\mathrm{d}X - \int_{\partial K_p^*} a(X,u)\nabla u \cdot n_P \mathrm{d}s = \int_{K_p^*} f(X,t,u)\mathrm{d}X, \quad P \in N_h, \tag{2.2}$$

where $n_P$ is a unit outer normal vectors to the involved integral domain.

Let $S_h^*$ be the piecewise constant space associated with the dual mesh $T_h^*$. Multiplying (2.2) by a test function $v \in S_h^*$ and summing over $N_h$ yields

$$\sum_{P\in N_h} v(P)\int_{K_p^*} \psi(X,u)\frac{\partial u}{\partial \tau}\,\mathrm{d}X - \sum_{P\in N_h} v(P)\int_{\partial K_p^*} a(X,u)\nabla u \cdot n_P \mathrm{d}s = \sum_{P\in N_h} v(P)\int_{K_p^*} f(X,t,u)\,\mathrm{d}X. \tag{2.3}$$

In order to write (2.3) in a more compact form, we shall introduce some bilinear forms. Let $K_{ij} = [x_i,\ x_{i+1}] \times [y_j,\ y_{j+1}]$ $(0 \leqslant i < n_x,\ 0 \leqslant j < n_y)$ be a rectangle (see Fig. 1) with the barycenter $O$ and the middle points $M_k$ $(1 \leqslant k \leqslant 4)$ of four edges (in Fig. 1 the dotted lines denote the interface of the corresponding control volumes). Regroup the summation to see that

$$\begin{aligned}\sum_{P\in N_h} v(P)\int_{\partial K_p^*} a(X,u)\nabla u \cdot n_P\,\mathrm{d}s = \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1}\Bigg\{\sum_{k=1}^{4}\Bigg[&v(P_k)\int_{\partial K_{P_k}^*\cap K_{ij}} a(X,u)\nabla u\cdot n_{P_k}\,\mathrm{d}s \\ &+ v(M_k)\int_{\partial K_{M_k}^*\cap K_{ij}} a(X,u)\nabla u\cdot n_{M_k}\,\mathrm{d}s\Bigg] \\ &+ v(O)\int_{\partial K_O^*\cap K_{ij}} a(X,u)\nabla u\cdot n_O\,\mathrm{d}s\Bigg\}.\end{aligned}$$

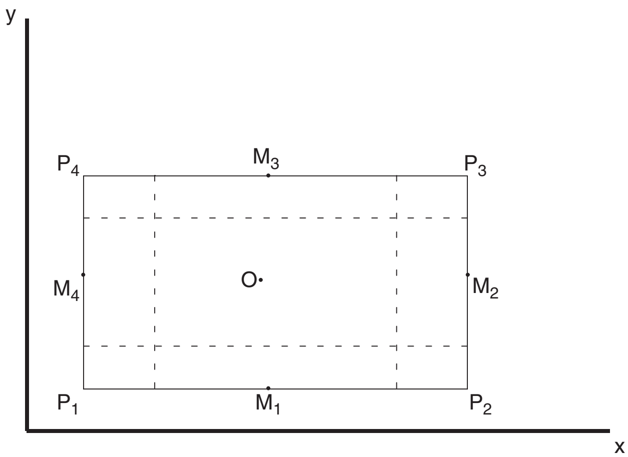


Fig. 1. Rectangle $K_{ij}$.

Then we define $A_{ij}(w; u, v)$ as follows:

$$A_{ij}(w; u, v) = -\sum_{k=1}^{4}\left[v(P_k)\int_{\partial K^*_{P_k}\cap K_{ij}} a(X, w)\nabla u \cdot n_{P_k}\,\mathrm{d}s + v(M_k)\int_{\partial K^*_{M_k}\cap K_{ij}} a(X, w)\nabla u \cdot n_{M_k}\,\mathrm{d}s\right]$$
$$- v(O)\int_{\partial K^*_{O}\cap K_{ij}} a(X, w)\nabla u \cdot n_{O}\,\mathrm{d}s, \quad (u, v) \in H^1_0(\Omega) \times S^*_h. \tag{2.4}$$

If $a(x, w) \equiv 1$, we use $A_{ij}(u, v)$ to denote $A_{ij}(w; u, v)$.

Let $A(u, v) = \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y 1} A_{ij}(u, v)$ and $A(w; u, v) = \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1} A_{ij}(w; u, v)$. Now it is more convenient to rewrite (2.3) in the following form:

$$\left(\psi(u)\frac{\partial u}{\partial \tau}, v\right) + A(u; u, v) = (f(u), v), \quad v \in S^*_h. \tag{2.5}$$

In the papers below, the usual Sobolev notation $W^{k,p}(\Omega)$, $H^k = W^{k,2}(\Omega)$, $L^p = W^{0,p}(\Omega)$ and $W^{k,p}(0, T; X)$ will be used, where $X$ is a Sobolev space, $k \geqslant 1$, $1 \leqslant p \leqslant \infty$. We denote by $\|\cdot\|_X$ the norm of $X$. For brevity, we will drop $\Omega$ and $0, T$ in some cases. For example, $\|\cdot\|_{L^2(L^2)}$ will denote $\|\cdot\|_{L^2(0,T;L^2(\Omega))}$.

Let $S_h \subset H^1_0(\Omega)$ be the standard biquadratic finite element space associated with the primal mesh $T_h$. We introduce the following two operators $\Pi$ and $\Pi^*$, where $\Pi : H^1_0(\Omega) \to S_h$ is an interpolation operator satisfying [4]

$$\|u - \Pi u\|_{H^k} \leqslant Mh^{3-k}\|u\|_{H^3}, \quad k \leqslant 3, \quad u \in H^1_0(\Omega) \cap H^3(\Omega) \tag{2.6}$$

and $\Pi^* : S_h \to S^*_h$ is a piecewise constant interpolation operator.

To study features of the discrete scheme of (2.5), we shall define some discrete norms on $S_h$ such that

$$|||u_h|||_0^2 = (u_h, \Pi^* u_h), \quad |u_h|^2_{0,h} = (\Pi^* u_h, \Pi^* u_h), \quad |u_h|^2_{1,h} = \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1} |u_h|^2_{1,h,K_{ij}}, \tag{2.7}$$

where

$$|u_h|^2_{1,h,K_{ij}} = \sum_{k=1}^{4}[(u_h(M_k) - u_h(P_k))^2 + (u_h(P_{k+1}) - u_h(M_k))^2 + (u_h(O) - u_h(M_k))^2].$$

If $k = 4$, we set $P_{k+1} = P_1$.

## 3. Some auxiliary results

In this section, we will study the propositions of the inner product $(u_h, \Pi^* v_h)$, the bilinear form $A(u_h, \Pi^* v_h)$ and $A(w; u_h, \Pi^* v_h)$. First, the following three lemmas indicate that the discrete norms defined in (2.7) are equivalent to the corresponding $L^2$-norm or $H^1$-seminorm.

**Lemma 1.** *There exist two positive constants $M_0$ and $M_1$ independent of $h$, such that*

$$M_0\|u_h\|_{L^2} \leqslant |||u_h|||_0 \leqslant M_1\|u_h\|_{L^2}, \quad u_h \in S_h. \tag{3.1}$$

**Proof.** We define a vector $\mathscr{X}_{u_h} \in R^9$ on the rectangle $K_{ij}$ (See Fig. 1):

$$\mathscr{X}_{u_h} = [u_h(P_1), u_h(M_4), u_h(P_4), u_h(M_1), u_h(O), u_h(M_3), u_h(P_2), u_h(M_2), u_h(P_3)]^{\mathrm{T}},$$

and two symmetric positive definite matrices $G$ and $\widehat{G}$:

$$G = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \widehat{G} = \begin{bmatrix} \frac{83}{648} & \frac{4}{81} & -\frac{7}{648} \\ \frac{4}{81} & \frac{46}{81} & \frac{4}{81} \\ -\frac{7}{648} & \frac{4}{81} & \frac{83}{648} \end{bmatrix}.$$

Using the tensor product basis, a direct calculation shows that

$$\|u_h\|_{L^2}^2 = \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} (\mathscr{X}_{u_h}^{\mathrm{T}} G \otimes G \mathscr{X}_{u_h} h_{x,i+1} h_{y,j+1}),$$

$$|||u_h|||_0^2 = \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} (\mathscr{X}_{u_h}^{\mathrm{T}} \widehat{G} \otimes \widehat{G} \mathscr{X}_{u_h} h_{x,i+1} h_{y,j+1}).$$

From the properties of tensor product, we know that $G \otimes G$ and $\widehat{G} \otimes \widehat{G}$ are all symmetric positive definite. So there exist two positive constants $M_0$ and $M_1$, only depending upon the eigenvalues of the matrices $G$ and $\widehat{G}$, such that

$$M_0\|u_h\|_{L^2} \leqslant |||u_h|||_0 \leqslant M_1\|u_h\|_{L^2}. \qquad \square$$

**Lemma 2.** *There exist two positive constants $M_0$ and $M_1$ independent of $h$, such that*

$$M_0\|u_h\|_{L^2} \leqslant |u_h|_{0,h} \leqslant M_1\|u_h\|_{L^2}, \quad u_h \in S_h. \tag{3.2}$$

**Proof.** A similar argument as in Lemma 1 gives the desired result. $\square$

**Lemma 3.** *There exist positive constants $M_0$ and $M_1$ independent of $h$ such that*

$$M_0|u_h|_{H^1} \leqslant |u_h|_{1,h} \leqslant M_1|u_h|_{H^1}, \quad u_h \in S_h. \tag{3.3}$$

**Proof.** For $u_h \in S_h$, we define two vectors $\mathscr{Y}_{u_h}$ and $\mathscr{Z}_{u_h}$ on $K_{ij}$, where

$$\begin{aligned} \mathscr{Y}_{u_h} = [&u_h(M_1) - u_h(P_1), u_h(O) - u_h(M_4), u_h(M_3) - u_h(P_4), \\ &u_h(P_2) - u_h(M_1), u_h(M_2) - u_h(O), u_h(P_3) - u_h(M_3)]^{\mathrm{T}}, \\ \mathscr{Z}_{u_h} = [&u_h(M_4) - u_h(P_1), u_h(P_4) - u_h(M_4), u_h(O) - u_h(M_1), \\ &u_h(M_3) - u_h(O), u_h(M_2) - u_h(P_2), u_h(P_3) - u_h(M_2)]^{\mathrm{T}}. \end{aligned}$$

By a direct calculation, we obtain

$$|u_h|^2_{H^1(K_{ij})} = \mathscr{Y}^{\mathrm{T}}_{u_h} H \otimes G \mathscr{Y}_{u_h} \frac{h_{y,j+1}}{h_{x,i+1}} + \mathscr{Z}^{\mathrm{T}}_{u_h} G \otimes H \mathscr{Z}_{u_h} \frac{h_{x,i+1}}{h_{y,j+1}},$$

where $G$ is given in Lemma 1 and

$$H = \begin{bmatrix} 7/3 & -1/3 \\ -1/3 & 7/3 \end{bmatrix}.$$

Since the matrices $H \otimes G$ and $G \otimes H$ are all positive definite, then it follows from the regularity condition (2.1) that

$$M_0\mu(\mathscr{Y}^{\mathrm{T}}_{u_h}\mathscr{Y}_{u_h} + \mathscr{Z}^{\mathrm{T}}_{u_h}\mathscr{Z}_{u_h}) \leqslant |u_h|^2_{H^1(K_{ij})} \leqslant \frac{M_1}{\mu}(\mathscr{Y}^{\mathrm{T}}_{u_h}\mathscr{Y}_{u_h} + \mathscr{Z}^{\mathrm{T}}_{u_h}\mathscr{Z}_{u_h}), \tag{3.4}$$

where the constants $M_0, M_1 > 0$ depending upon the eigenvalues of the matrices $G$ and $H$.

Note that $\mathscr{Y}^{\mathrm{T}}_{u_h}\mathscr{Y}_{u_h} + \mathscr{Z}^{\mathrm{T}}_{u_h}\mathscr{Z}_{u_h} = |u_h|^2_{1,h,K_{ij}}$. Summing (3.4) over all $K_{ij}$, we complete the proof. □

By the calculation in Lemma 1, we can verify that $(u_h, \Pi^* v_h) = \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1}(\mathscr{X}^{\mathrm{T}}_{v_h}\widehat{G}\otimes\widehat{G}\mathscr{X}_{u_h})h_{x,i+1}h_{y,j+1}$. Thus, from the symmetry of the matrix $\widehat{G}$, we obtain the following lemma.

**Lemma 4.** *For any $u_h, v_h \in S_h$, we have*

$$(u_h, \Pi^* v_h) = (v_h, \Pi^* u_h). \tag{3.5}$$

In the lemmas below, we shall state some key features of the bilinear form $A(u_h, \Pi^* v_h)$ and $A(w; u_h, \Pi^* v_h)$.

**Lemma 5.** *For any $u_h, v_h \in S_h$, there exists positive constants $\gamma$ and $M$ independent of $h$ such that*

$$A(u_h, \Pi^* v_h) = A(v_h, \Pi^* u_h), \tag{3.6}$$

$$A(u_h, \Pi^* u_h) \geqslant \gamma \|u_h\|^2_{H^1}, \tag{3.7}$$

$$|A(u_h, \Pi^* v_h)| \leqslant M \|u_h\|_{H^1}\|v_h\|_{H^1}. \tag{3.8}$$

**Proof.** For any $K_{ij}$, by the definition (2.4) and the tensor product basis, we calculate to obtain

$$A_{ij}(u_h, \Pi^* v_h) = \mathscr{Y}^{\mathrm{T}}_{v_h} H \otimes \widehat{G} \mathscr{Y}_{u_h} \frac{h_{y,j+1}}{h_{x,i+1}} + \mathscr{Z}^{\mathrm{T}}_{v_h} \widehat{G} \otimes H \mathscr{Z}_{u_h} \frac{h_{x,i+1}}{h_{y,j+1}}. \tag{3.9}$$

Here the matrices $H$, $\widehat{G}$ and the vector $\mathscr{Y}_{v_h}$, $\mathscr{Z}_{v_h}$ are given in Lemmas 1 and 2.

By the properties of tensor product, it is easy to see that $H \otimes \widehat{G}$ and $\widehat{G} \otimes H$ are all symmetric and positive definite. Thus, $A_{ij}(u_h, \Pi^* v_h) = A_{ij}(v_h, \Pi^* u_h)$. Gather the result over all $K_{ij}$ to get (3.6).

Choosing $v_h = u_h$ in (3.9) and using the condition (2.1) yields

$$A_{ij}(u_h, \Pi^* u_h) \geqslant \mu M_0(\mathscr{Y}^{\mathrm{T}}_{u_h}\mathscr{Y}_{u_h} + \mathscr{Z}^{\mathrm{T}}_{u_h}\mathscr{Z}_{u_h}) = \mu M_0 |u_h|^2_{1,h,K_{ij}},$$

where $M_0$ is a positive constant depending on the minimal eigenvalues of $H$ and $\widehat{G}$. Then it follows from Lemma 3 and Poincaré inequality that

$$A(u_h, \Pi^* u_h) \geqslant \mu M_0 |u_h|^2_{1,h} \geqslant \frac{\mu M_0}{M_1^2}|u_h|^2_{H^1} \geqslant \gamma \|u_h\|^2_{H^1}.$$

At last, by (2.1) and the Cauchy–Schwartz inequality, we have

$$A_{ij}(u_h, \Pi^* v_h) \leqslant \frac{M_1}{\mu}(\mathscr{Y}^{\mathrm{T}}_{v_h}\mathscr{Y}_{u_h} + \mathscr{Z}^{\mathrm{T}}_{v_h}\mathscr{Z}_{u_h}) \leqslant \frac{M_1}{\mu}|u_h|_{1,h,K_{ij}}|v_h|_{1,h,K_{ij}},$$

where $M_1$ is a positive constant depending on the maximal eigenvalues of $H$ and $\widehat{G}$. Then apply Lemma 3 to obtain

$$A(u_h, \Pi^* v_h) \leqslant \frac{M_1}{\mu} \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} |u_h|_{1,h,K_{ij}} |v_h|_{1,h,K_{ij}} \leqslant \frac{M_1}{\mu} |u_h|_{1,h} |v_h|_{1,h} \leqslant M \|u_h\|_{H^1} \|v_h\|_{H^1}. \qquad \square$$

**Lemma 6.** *For any $u_h, v_h \in S_h$ and $u \in H^3(\Omega)$,*

$$|A(w; u - u_h, \Pi^* v_h)| \leqslant M(h^2 \|u\|_{H^3} + |u - u_h|_{H^1}) \|v_h\|_{H^1}. \tag{3.10}$$

**Proof.** Reordering by edges, we get

$$\begin{aligned} |A_{ij}(w; u - u_h, \Pi^* v_h)| = \Bigg| &\sum_{k=1}^{4} (v_h(M_k) - v_h(P_k)) \int_{\partial K^*_{M_k} \cap \partial K^*_{P_k}} a(X, w) \nabla(u - u_h) \cdot n_{P_k} \, \mathrm{d}s \\ &+ \sum_{k=1}^{4} (v_h(P_{k+1}) - v_h(M_k)) \int_{\partial K^*_{P_{k+1}} \cap \partial K^*_{M_k}} a(X, w) \nabla(u - u_h) \cdot n_{M_l} \, \mathrm{d}s \\ &+ \sum_{l=k}^{4} (v_h(O) - v_h(M_k)) \int_{\partial K^*_O \cap \partial K^*_{M_k}} a(X, w) \nabla(u - u_h) \cdot n_{M_k} \mathrm{d}s \Bigg|. \end{aligned}$$

Using the Cauchy–Schwartz inequality and the trace theorem yields

$$\begin{aligned} \left| \int_{\partial K^*_{M_l} \cap \partial K^*_{P_l}} a(X, w) \nabla(u - u_h) \cdot n_{P_l} \, \mathrm{d}s \right| &\leqslant M a^* \|u - u_h\|_{H^1(\partial K^*_{M_l} \cap \partial K^*_{P_l})} h^{1/2} \\ &\leqslant M a^* \|u - u_h\|^{1/2}_{H^1(K_{ij})} \|u - u_h\|^{1/2}_{H^2(K_{ij})} h^{1/2}. \end{aligned}$$

From the triangle inequality and the inverse estimates, we have

$$\begin{aligned} \|u - u_h\|_{H^2(K_{ij})} &\leqslant M(h\|u\|_{H^3(K_{ij})} + h^{-1} \|\Pi u - u_h\|_{H^1(K_{ij})}) \\ &\leqslant M(h\|u\|_{H^3(K_{ij})} + h^{-1} \|u - u_h\|_{H^1(K_{ij})}). \end{aligned}$$

Hence, combining above three estimates gives

$$\left| \int_{\partial K^*_{M_l} \cap \partial K^*_{P_l}} a(X, w) \nabla(u - u_h) \cdot n_{P_l} \mathrm{d}s \right| \leqslant M(h^2 \|u\|_{H^3(K_{ij})} + \|u - u_h\|_{H^1(K_{ij})}).$$

It is obvious that $|v_h(M_l) - v_h(P_l)| \leqslant |v_h|_{1,h,K_{ij}}$. Estimating the remaining terms similarly, we obtain

$$|A_{ij}(w; u - u_h, \Pi^* v_h)| \leqslant M(h^2 \|u\|_{H^3(K_{ij})} + \|u - u_h\|_{H^1(K_{ij})}) |v_h|_{1,h,K_{ij}}.$$

Then sum the result over $K_{ij}$ and use the Cauchy–Schwartz inequality and Lemma 2 to get

$$\begin{aligned} |A(w; u - u_h, \Pi^* v_h)| &\leqslant M(h^2 \|u\|_{H^3} + \|u - u_h\|_{H^1}) |v_h|_{1,h} \\ &\leqslant M(h^2 \|u\|_{H^3} + \|u - u_h\|_{H^1}) \|v_h\|_{H^1}. \qquad \square \end{aligned}$$

**Lemma 7.** *Suppose that $u \in W^{1,\infty}(\Omega)$ and $w \in H^3$. For any $v_h, w_h \in S_h$,*

$$|A(w; u, \Pi^* v_h) - A(w_h; u, \Pi^* v_h)| \leqslant M(h^3 \|w\|_{H^3} + \|w - w_h\|_{L^2}) \|u\|_{W^{1,\infty}} \|v_h\|_{H^1}.$$

**Proof.** Reordering by edges, we have,

$$\begin{aligned}
&|A_{ij}(w; u, \Pi^* v_h) - A_{ij}(w_h; u, \Pi^* v_h)| \\
&\quad = \Bigg| \sum_{k=1}^{4} (v_h(M_k) - v_h(P_k)) \int_{\partial K^*_{M_k} \cap \partial K^*_{P_k}} (a(X, w) - a(X, w_h)) \nabla u \cdot n_{P_k} \, \mathrm{d}s \\
&\qquad + \sum_{k=1}^{4} (v_h(P_{k+1}) - v_h(M_k)) \int_{\partial K^*_{P_{k+1}} \cap \partial K^*_{M_k}} (a(X, w) - a(X, w_h)) \nabla u \cdot n_{M_k} \, \mathrm{d}s \\
&\qquad + \sum_{k=1}^{4} (v_h(O) - v_h(M_k)) \int_{\partial K^*_{O} \cap \partial K^*_{M_k}} (a(X, w) - a(X, w_h)) \nabla u \cdot n_{M_k} \, \mathrm{d}s \Bigg|.
\end{aligned}$$

Since $|(\partial/\partial u) a(X, u)| \leqslant a^*$, then $|a(X, w) - a(X, w_h)| \leqslant a^* |w - w_h|$. Therefore, we know from the argument in Lemma 6 that

$$\begin{aligned}
\left| \int_{\partial K^*_{M_k} \cap \partial K^*_{P_k}} (a(X, w) - a(X, w_h)) \nabla u \cdot n_{P_k} \, \mathrm{d}s \right| &\leqslant a^* \|u\|_{W^{1,\infty}} \int_{\partial K^*_{M_l} \cap \partial K^*_{P_l}} |w - w_h| \, \mathrm{d}s \\
&\leqslant M a^* \|u\|_{W^{1,\infty}} (h^3 \|w\|_{H^3(K_{ij})} + \|w - w_h\|_{L^2(K_{ij})}).
\end{aligned}$$

Then, proceeding similarly as in Lemma 6 yields the desired result. □

**Lemma 8.** *Suppose that $w \in W^{1,\infty}(\Omega)$. For any $u_h \in S_h$, when $h$ is small enough, there exists a constant $\gamma > 0$ such that*

$$A(w; u_h, \Pi^* u_h) \geqslant (\gamma - Mh(1 + \|w\|_{W^{1,\infty}})) \|u_h\|^2_{H^1}. \tag{3.11}$$

**Proof.** Let $a(O)$ denote the value of $a(X, w)$ at the barycenter of $K_{ij}$. Then

$$A_{ij}(w; u_h, \Pi^* u_h) = a(O) A_{ij}(u_h, \Pi^* u_h) + (A_{ij}(w; u_h, \Pi^* u_h) - a(O) A_{ij}(u_h, \Pi^* u_h)).$$

It follows from Lemma 5 that

$$\sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} a(O) A_{ij}(u_h, \Pi^* u_h) \geqslant M_0 \mu a_* \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} |u_h|^2_{1,h,K_{ij}} \geqslant \gamma \|u_h\|^2_{H^1}.$$

Taking $h$ small enough such that $|X - X_O| \leqslant \varepsilon_0$, then by Assumption 1,

$$|a(X, w) - a(O)| \leqslant L|X - X_O| + a^* |w - w(O)| \leqslant Mh(1 + \|w\|_{W^{1,\infty}}).$$

Thus, a similar argument as in Lemma 6 gives

$$\begin{aligned}
\sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} |A_{ij}(w; u_h, \Pi^* u_h) - a(O) A_{ij}(u_h, \Pi^* u_h)| &\leqslant Mh(1 + \|w\|_{W^{1,\infty}}) \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} \|u_h\|^2_{H^1(K_{ij})} \\
&= Mh(1 + \|w\|_{W^{1,\infty}}) \|u_h\|^2_{H^1}.
\end{aligned}$$

Finally, the proof is completed by combining the above estimates. □

**Lemma 9.** *Suppose that $w \in W^{1,\infty}(\Omega)$. For any $u_h, v_h \in S_h$, when $h$ is small enough,*

$$|A(w; u_h, \Pi^* v_h) - A(w; v_h, \Pi^* u_h)| \leqslant Mh(1 + \|w\|_{W^{1,\infty}})\|u_h\|_{H^1}\|v_h\|_{H^1}. \tag{3.12}$$

**Proof.** We still let $a(O)$ denote the value of $a(x, w)$ at the barycenter of $K_{ij}$, then

$$A_{ij}(w; u_h, \Pi^* v_h) = a(O)A_{ij}(u_h, \Pi^* v_h) + [A_{ij}(w; u_h, \Pi^* v_h) - a(O)A_{ij}(u_h, \Pi^* v_h)].$$

For sufficiently small $h$, a similar argument as in Lemma 8 yields

$$\left|\sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1}(A_{ij}(w; u_h, \Pi^* v_h) - a(O)A_{ij}(u_h, \Pi^* v_h))\right| \leqslant Mh(1 + \|w\|_{W^{1,\infty}})\|u_h\|_{H^1}\|v_h\|_{H^1}.$$

By the result in Lemma 3, we have $A_{ij}(u_h, \Pi^* v_h) - A_{ij}(v_h, \Pi^* u_h) = 0$.

Thus, it follows from the above estimates and the triangle inequality that

$$\begin{aligned}|A(w; u_h, \Pi^* v_h) - A(w; v_h, \Pi^* u_h)| &\leqslant \left|\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[A_{ij}(w; u_h, \Pi^* v_h) - a(O)A_{ij}(u_h, \Pi^* v_h)\right| \\ &\quad + \left|\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[A_{ij}(w; v_h, \Pi^* u_h) - a(O)A_{ij}(v_h, \Pi^* u_h)]\right| \\ &\leqslant Mh(1 + \|w\|_{W^{1,\infty}})\|u_h\|_{H^1}\|v_h\|_{H^1}.\end{aligned}$$

At the end of this section, we consider the relationship between the bilinear form $A(u_h, \Pi^* v_h)$ and $A(w; u_h, \Pi^* v_h)$. □

**Lemma 10.** *Suppose that $\rho - M_0 a^* \geqslant 0$, where $\rho$ and $M_0$ are constants, and $a^*$ is the constant given in Assumption 1. For any $u_h \in S_h$, when $h$ is sufficiently small, there exists a positive constant $\gamma$ such that*

$$\rho A(u_h, \Pi^* u_h) - M_0 A(w; u_h, \Pi^* u_h) \geqslant -Mh(1 + \|w\|_{W^{1,\infty}})\|u_h\|_{H^1}^2. \tag{3.13}$$

**Proof.** First, note that $\lambda - M_0 a(O) \geqslant \lambda - M_0 a^* \geqslant 0$. Then from Lemma 5, we have

$$\sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1}(\rho - M_0 a(O))A_{ij}(u_h, \Pi^* u_h) \geqslant (\rho - M_0 a^*)A(u_h, \Pi^* u_h) \geqslant 0.$$

The argument in Lemma 8 shows that

$$M_0 \sum_{i=0}^{n_x-1}\sum_{j=0}^{n_y-1}|A_{ij}(w; u_h, \Pi^* u_h) - a(O)A_{ij}(u_h, \Pi^* u_h)| \leqslant Mh(1 + \|w\|_{W^{1,\infty}})\|u_h\|_{H^1}^2.$$

Therefore, using the above two estimates and the triangle inequality gives the desired result. □

## 4. FVE scheme and error estimates

In this section we shall first introduce a symmetric FVE scheme for the problem (1.1). For any positive integer $N$, let $\Delta t = T/N$, $t_n = n\Delta t$ $(n \leqslant N)$. Let

$$u^n = u(t_n), \quad \delta u^n = u^n - u^{n-1}, \quad \delta^2 u^n = \delta(\delta u^n), \quad \partial_t u^n = \frac{u^n - u^{n-1}}{\Delta t}.$$

Moreover, we set $\partial_t u^0 = 0$. Let $E$ be an extrapolator satisfying:

$$Eu^{n+1} = \begin{cases} 2u^n - u^{n-1}, & n \geqslant 1, \\ u^0, & n = 0. \end{cases} \tag{4.1}$$

According to Lemmas 4, 5 and 9, we can see that $(u_h, \Pi^* v_h)$ and $A(u_h, \Pi^* v_h)$ are symmetric, while $A(w; u_h, \Pi^* v_h)$ is not. Then in order to construct a symmetric FVE scheme, we hope that the discrete form of $A(u; u, v)$ in (2.5) is known at each time level. So our FVE scheme is defined by: find $u_h^{n+1} \in S_h$, $1 \leqslant n \leqslant N-1$ such that

$$\begin{aligned} &\left(\frac{u_h^{n+1} - \hat{u}_h^n}{\Delta t}, v\right) + \rho A(\delta^2 u_h^{n+1}, v) + \rho^2 \Delta t B(\delta^2 u_h^{n+1}, v) \\ &\quad = \frac{1}{3}\left(\frac{\hat{u}_h^n - \hat{\hat{u}}_h^{n-1}}{\Delta t}, v\right) + \frac{2}{3}\left(f^{n+1}(Eu_h^{n+1}), v\right) - \frac{2}{3} A(Eu_h^{n+1}; Eu_h^{n+1}, v), \quad v \in S_h^*, \end{aligned} \tag{4.2}$$

where $\hat{u}_h^n = u_h^n(\hat{X}) = u_h(X - b(X, Eu_h^{n+1})\Delta t, t_n)$, $\hat{\hat{u}}_h^{n-1} = u_h^{n-1}(\hat{\hat{X}}) = u_h(X - 2b(X, Eu_h^{n+1})\Delta t, t_{n-1})$ and $B(\cdot,\cdot)$ is a symmetric splitting perturbation to be defined below.

The initial approximations $\{u_h^0, u_h^1\}$ can be obtained by

$$u_h^0 = \Pi u_0, \tag{4.3}$$

$$\left(\frac{u_h^1 - \hat{u}_h^0}{\Delta t}, v\right) + \rho A(\delta u_h^1, v) + \rho^2 \Delta t B(\delta u_h^1, v) = (f^1(u_h^0), v) - A(u_h^0; u_h^0, v), \quad v \in S_h^*. \tag{4.4}$$

In the scheme above, the constant $\rho \geqslant \frac{2}{3} a^*$ and $\Delta t < 1/2L$, where $L$ is the Lipschitz constant given in Assumption 1.

**Remark 3.** The main idea to construct scheme (4.2)–(4.4) is motivated by [10,9,13], where similar schemes based on finite element methods were derived. In those paper, the stabilized term $B(\cdot,\cdot)$ is decided by the coefficient matrices of $(\cdot,\cdot)$ and $A(\cdot,\cdot)$ such that the discrete scheme can be factorized into some low dimensional equations and solved by alternating-direction. This is also valid for our method.

According to Lemmas 4 and 5, it is easy to see that the coefficient matrix of the scheme above is symmetric, positive definite, and time–independent. More important, since this scheme can be solved by alternating–direction, it can be executed in parallel computers and is more efficient than general ones [10,19,9,13].

**Remark 4.** We point out that if $\Delta t < 1/2L$ and $h$ is small enough, then for any $X = (x, y) \in \Omega$, the corresponding $\hat{X}, \hat{\hat{X}} \in \Omega$. Without loss of generality, we only prove that $\hat{\hat{X}} \in \Omega$. If $b_1(X, Eu_h^{n+1}) \geqslant 0$, it is obvious that $\hat{\hat{x}} < b_x$. Let $Y = (a_x, y) \in \partial\Omega$. Then for sufficiently small $h$, by Assumption 1, we have $\hat{\hat{x}} - a_x = x - a_x - 2[b_1(X, Eu_h^{n+1}) - b_1(Y, Eu_h^{n+1})]\Delta t \geqslant x - a_x - 2L\Delta t|x - a_x| > 0$. Hence, $\hat{\hat{x}} > a_x$. If $b_1(X, Eu_h^{n+1}) < 0$, the same result holds. Consequently, we also can prove that $a_y < \hat{\hat{y}} < b_y$.

Now we will construct the perturbation term $B$ by the matrices of $(u_h, \Pi^* v_h)$ and $A(u_h, \Pi^* v_h)$. From Lemmas 4 and 5, the coefficient matrix of $(u_h, \Pi^* v_h)$ is $C_x \otimes C_y$ and the coefficient matrix of $A(u_h, \Pi^* v_h)$ is $(C_x \otimes A_y + A_x \otimes C_y)$, where for $\lambda = x$ or $y$

$$C_\lambda=\begin{bmatrix}
\frac{46}{81}h_{\lambda,1} & \frac{4}{81}h_{\lambda,1} & & & & & & \\
\frac{4}{81}h_{\lambda,1} & \frac{83}{648}(h_{\lambda,1}+h_{\lambda,2}) & \frac{4}{81}h_{\lambda,2} & -\frac{7}{648}h_{\lambda,2} & & & & \\
 & \frac{4}{81}h_{\lambda,2} & \frac{46}{81}h_{\lambda,2} & \frac{4}{81}h_{\lambda,2} & & & & \\
 & -\frac{7}{648}h_{\lambda,2} & \frac{4}{81}h_{\lambda,2} & \frac{83}{648}(h_{\lambda,2}+h_{\lambda,3}) & \frac{4}{81}h_{\lambda,3} & -\frac{7}{648}h_{\lambda,3} & & \\
 & & & \frac{4}{81}h_{\lambda,3} & \frac{46}{81}h_{\lambda,3} & \frac{4}{81}h_{\lambda,3} & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & \frac{4}{81}h_{\lambda,n_\lambda} & \frac{46}{81}h_{\lambda,n_\lambda}
\end{bmatrix}$$

and

$$A_\lambda=\begin{bmatrix}
\frac{16}{3h_{\lambda,1}} & -\frac{8}{3h_{\lambda,1}} & & & & & & \\
-\frac{8}{3h_{\lambda,1}} & \left(\frac{7}{3h_{\lambda,1}}+\frac{7}{3h_{\lambda,2}}\right) & -\frac{8}{3h_{\lambda,2}} & \frac{1}{3h_{\lambda,2}} & & & & \\
 & -\frac{8}{3h_{\lambda,2}} & \frac{16}{3h_{\lambda,2}} & -\frac{8}{3h_{\lambda,2}} & & & & \\
 & \frac{1}{3h_{\lambda,2}} & -\frac{8}{3h_{\lambda,2}} & \left(\frac{7}{3h_{\lambda,2}}+\frac{7}{3h_{\lambda,3}}\right) & -\frac{8}{3h_{\lambda,3}} & \frac{1}{3h_{\lambda,3}} & & \\
 & & & -\frac{8}{3h_{\lambda,3}} & \frac{16}{3h_{\lambda,3}} & -\frac{8}{3h_{\lambda,3}} & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & -\frac{8}{3h_{\lambda,n_\lambda}} & \frac{16}{3h_{\lambda,n_\lambda}}
\end{bmatrix}.$$

Thus, for $u_h \in S_h$, $v \in S_h^*$, we define

$$B(u_h, v) = \mathscr{V}^{\mathrm{T}} A_x \otimes A_y \, \mathscr{U}_h, \tag{4.5}$$

where $\mathscr{V}$ is a column vector satisfying $\mathscr{V}_{(2n_x-1)(i-1)+j} = v(x_{i/2}, y_{j/2})$, $\quad 1 \leqslant i \leqslant 2n_x - 1$, $1 \leqslant j \leqslant 2n_y - 1$. The vector $\mathscr{U}_h$ is defined similarly. It is easy to verify that

$$B(u_h, \Pi^* v_h) = \left(\frac{\partial^2 u_h}{\partial x \partial y}, \frac{\partial^2 v_h}{\partial x \partial y}\right), \quad u_h, v_h \in S_h. \tag{4.6}$$

Next, we shall derive some useful prior estimates to be used in the error analysis.

**Lemma 11.** *Suppose that $w^{n+1} \in W^{1,\infty}(\Omega)$ and $\partial_t w^{n+1} \in L^\infty(\Omega)$, $n < R$. If $h = O(\Delta t)$, then for sufficiently small $h$,*

$$\sum_{n=1}^{R-1} A(w^{n+1}; u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \geqslant \frac{1}{2} A(w^R; u_h^R, \Pi^* u_h^R) - M(\Delta t \sum_{n=1}^{R-1} \|u_h^{n+1}\|_{H^1}^2 + \|u_h^1\|_{H^1}^2), \quad u_h \in S_h. \tag{4.7}$$

**Proof.** Note that

$$\begin{aligned} &A(w^{n+1}; u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \\ &\quad = \tfrac{1}{2}(A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^{n+1}) - A(w^n; u_h^n, \Pi^* u_h^n)) \\ &\qquad - \tfrac{1}{2}(A(w^{n+1}; u_h^n, \Pi^* u_h^n) - A(w^n; u_h^n, \Pi^* u_h^n)) \\ &\qquad + \tfrac{1}{2}(A(w^{n+1}; u_h^n, \Pi^* u_h^{n+1}) - A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^n)) \\ &\qquad + \tfrac{1}{2}\, A(w^{n+1}; \delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})). \end{aligned} \tag{4.8}$$

We now estimate the terms on the right hand of (4.8) sequently. First, using Lemma 6, we have

$$\begin{aligned} \sum_{n=1}^{R-1}(A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^{n+1}) - A(w^n; u_h^n, \Pi^* u_h^n)) &= A(w^R; u_h^R, \Pi^* u_h^R) - A(w^1; u_h^1, \Pi^* u_h^1) \\ &\geqslant A(w^R; u_h^R, \Pi^* u_h^R) - M\|u_h^1\|_{H^1}^2. \end{aligned} \tag{4.9}$$

Second, when $\partial_t w^{n+1} \in L^\infty(\Omega)$ the argument in Lemma 8 gives

$$|A(w^{n+1}; u_h^n, \Pi^* u_h^n) - A(w^n; u_h^n, \Pi^* u_h^n)| \leqslant M(h + \Delta t)\|u_h^n\|_{H^1}^2. \tag{4.10}$$

Third, it follows from Lemma 9 that

$$|A(w^{n+1}; u_h^n, \Pi^* u_h^{n+1}) - A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^n)| \leqslant Mh\|u_h^n\|_{H^1}\|u_h^{n+1}\|_{H^1}. \tag{4.11}$$

At last, using Lemma 8 and taking $h$ small enough such that $Mh(1 + \|w^{n+1}\|_{W^{1,\infty}}) \leqslant \gamma$, then

$$A(w^{n+1}; \delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \geqslant (\gamma - Mh(1 + \|w^{n+1}\|_{W^{1,\infty}}))\|\delta u_h^{n+1}\|_{H^1}^2 \geqslant 0. \tag{4.12}$$

Therefore, combining (4.8)–(4.12) and using the Young inequality yields

$$\begin{aligned} &\sum_{n=1}^{R-1} A(w^{n+1}, u_h^{n+1}, \Pi^* \delta u_h^{n+1}) \\ &\quad \geqslant \frac{1}{2}\, A(w^R; u_h^R, \Pi^* u_h^R) - \frac{M}{2}\|u_h^1\|_{H^1}^2 - \frac{M}{2}(h + \Delta t)\sum_{n=1}^{R-1}(\|u_h^n\|_{H^1}^2 + \|u_h^n\|_{H^1}\|u_h^{n+1}\|_{H^1}) \\ &\quad \geqslant \frac{1}{2}\, A(w^R; u_h^R, \Pi^* u_h^R) - \frac{M}{2}\|u_h^1\|_{H^1}^2 - M(h + \Delta t)\sum_{n=1}^{R-1}(\|u_h^{n+1}\|_{H^1}^2 + \|u_h^n\|_{H^1}^2) \\ &\quad \geqslant \frac{1}{2}\, A(w^R; u_h^R, \Pi^* u_h^R) - \frac{M}{2}\|u_h^1\|_{H^1}^2 - 2M(h + \Delta t)\sum_{n=1}^{R-1} \|u_h^{n+1}\|_{H^1}^2 + \|u_h^1\|_{H^1}^2. \end{aligned}$$

Note that $h = O(\Delta t)$. Thus we get the desired result. $\square$

**Lemma 12.** *Suppose that* $w^{n+1} \in W^{1,\infty}(\Omega)$, $\partial_t w^{n+1} \in L^{\infty}(\Omega)$, $n < R$, *and* $\rho - M_0 a^* \geqslant 0$. If $h = O(\Delta t)$, *then for sufficiently small* $h$, *there exists a constant* $\gamma > 0$ *such that*

$$\sum_{n=1}^{R-1}(\rho A(\delta u_h^{n+1}, \Pi^* u_h^{n+1}) - M_0 A(w^{n+1}; \delta u_h^{n+1}, \Pi^* u_h^{n+1})) \\ \geqslant -M(\Delta t \sum_{n=1}^{R-1} \|u_h^{n+1}\|_{H^1}^2 + \Delta t \|u_h^1\|_{H^1}^2 + \|u_h^1\|_{H^1}^2), \quad u_h \in S_h. \tag{4.13}$$

**Proof.** We consider the terms on the left-hand side of (4.13). First, using Lemma 5 yields

$$\begin{aligned}
&\sum_{n=1}^{R-1} A(\delta u_h^{n+1}, \Pi^* u_h^{n+1}) \\
&= \frac{1}{2} \sum_{n=1}^{R-1}(A(u_h^{n+1}, \Pi^* u_h^{n+1}) - A(u_h^n, \Pi^* u_h^n)) + \frac{1}{2} \sum_{n=1}^{R-1} A(\delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \\
&= \frac{1}{2}(A(u_h^R, \Pi^* u_h^R) - A(u_h^1, \Pi^* u_h^1)) + \frac{1}{2} \sum_{n=1}^{R-1} A(\delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \\
&\geqslant \frac{1}{2} A(u_h^R, \Pi^* u_h^R) - M\|u_h^1\|_{H^1}^2 + \frac{1}{2} \sum_{n=1}^{R-1} A(\delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})).
\end{aligned} \tag{4.14}$$

Second, we see that

$$\begin{aligned}
A(w^{n+1}; \delta u_h^{n+1}, \Pi^* u_h^{n+1}) = &\tfrac{1}{2}(A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^{n+1}) - A(w^n; u_h^n, \Pi^* u_h^n)) \\
&- \tfrac{1}{2}(A(w^{n+1}; u_h^n, \Pi^* u_h^n) - A(w^n; u_h^n, \Pi^* u_h^n)) \\
&- \tfrac{1}{2}(A(w^{n+1}; u_h^n, \Pi^* u_h^{n+1}) - A(w^{n+1}; u_h^{n+1}, \Pi^* u_h^n)) \\
&+ \tfrac{1}{2} A(w^{n+1}; \delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})).
\end{aligned}$$

Then for sufficiently small $h$, proceeding a similar way as in Lemma 11 gives

$$\sum_{n=1}^{R-1} A(w^{n+1}; \delta u_h^{n+1}, \Pi^* u_h^{n+1}) \leqslant \frac{1}{2} A(w^R; u_h^R, \Pi^* u_h^R) + \frac{1}{2} \sum_{n=1}^{R-1} A(w^{n+1}; \delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) \\ + M\left(\Delta t \sum_{n=1}^{R-1} \|u_h^{n+1}\|_{H^1}^2 + \|u_h^1\|_{H^1}^2\right). \tag{4.15}$$

We now combine (4.14)–(4.15) to see that

$$\begin{aligned}
&\sum_{n=1}^{R-1}(\rho A(\delta u_h^{n+1}, \Pi^* u_h^{n+1}) - M_0 A(w^{n+1}; \delta u_h^{n+1}, \Pi^* u_h^{n+1})) \\
&\geqslant \frac{1}{2}(\rho A(u_h^R, \Pi^* u_h^R) - M_0 A(w^R; u_h^R, \Pi^* u_h^R)) \\
&\quad + \frac{1}{2} \sum_{n=1}^{R-1}(\rho A(\delta u_h^{n+1}, \Pi^*(\delta u_h^{n+1})) - M_0 A(w^{n+1}; \delta u_h^{n+1}, \Pi^* \delta u_h^{n+1})) \\
&\quad - M\left(\Delta t \sum_{n=1}^{R-1} \|u_h^{n+1}\|_{H^1}^2 + \|u_h^1\|_{H^1}^2\right).
\end{aligned} \tag{4.16}$$

Next we shall estimate the first two terms on the right-hand side of (4.16). According to Lemma 10, when $w^{n+1} \in W^{1,\infty}(\Omega)$, $n < R$,

$$\rho A(u_h^R, \Pi^* u_h^R) - M_0 A(w^R; u_h^R, \Pi^* u_h^R) \geqslant -Mh(1 + \|w^R\|_{W^{1,\infty}})\|u_h^R\|_{H^1}^2 \geqslant -Mh\|u_h^R\|_{H^1}^2, \tag{4.17}$$

and

$$\begin{aligned}&\rho A(\delta u_h^{n+1}, \Pi^* \delta u_h^{n+1}) - M_0 A(w^{n+1}; \delta u_h^{n+1}, \Pi^* \delta u_h^{n+1})\\ &\quad \geqslant -Mh(1 + \|w^{n+1}\|_{W^{1,\infty}})\|\delta u_h^{n+1}\|_{H^1}^2 \geqslant -Mh\|\delta u_h^{n+1}\|_{H^1}^2.\end{aligned} \tag{4.18}$$

Note that $\|\delta u_h^{n+1}\|_{H^1}^2 \leqslant 2(\|u_h^{n+1}\|_{H^1}^2 + \|u_h^n\|_{H^1}^2)$ and $h = O(\Delta t)$. Then, we combine (4.16)–(4.18) to have the desired result. □

**Lemma 13.** *Suppose that $\partial_t w^{n+1} \in L^\infty(\Omega)$, $n < R$, then for $u_h \in S_h$*

$$\begin{aligned}\Delta t \sum_{n=1}^{R-1} A(w^{n+1}; u^{n+1}, \Pi^* \partial_t u_h^{n+1}) \leqslant{}& A(w^R; u^R, \Pi^* u_h^R) - A(w^2; u^2, \Pi^* u_h^1)\\ &+ M\Delta t \sum_{n=2}^{R-1} (h^2[\|u^{n+1}\|_{H^3} + \|\partial_t u^{n+1}\|_{H^3}]\\ &+ \|\partial_t u^{n+1}\|_{H^1} + \|u^{n+1}\|_{H^1})\|u_h^n\|_{H^1}.\end{aligned} \tag{4.19}$$

**Proof.** We see that the left-hand side of (4.19) can be rewritten as follows:

$$\begin{aligned}&\Delta t \sum_{n=1}^{R-1} A(w^{n+1}; u^{n+1}, \Pi^* \partial_t u_h^{n+1})\\ &\quad = \Delta t \sum_{n=2}^{R-1} A(w^{n+1}; u^{n+1}, \Pi^* \partial_t u_h^{n+1}) + A(w^2; u^2, \Pi^* u_h^2) - A(w^2; u^2, \Pi^* u_h^1),\end{aligned}$$

where

$$\begin{aligned}&\Delta t \sum_{n=2}^{R-1} A(w^{n+1}; u^{n+1}, \Pi^* \partial_t u_h^{n+1})\\ &\quad = \sum_{n=2}^{R-1} (A(w^{n+1}; u^{n+1}, \Pi^* u_h^{n+1}) - A(w^{n+1}; u^{n+1}, \Pi^* u_h^n))\\ &\quad = \sum_{n=2}^{R-1} (A(w^{n+1}; u^{n+1}, \Pi^* u_h^{n+1}) - A(w^n; u^n, \Pi^* u_h^n))\\ &\qquad - \sum_{n=2}^{R-1} (A(w^{n+1}; u^{n+1}, \Pi^* u_h^n) - A(w^n; u^{n+1}, \Pi^* u_h^n))\\ &\qquad - \sum_{n=2}^{R-1} (A(w^n; u^{n+1}, \Pi^* u_h^n) - A(w^n; u^n, \Pi^* u_h^n)).\end{aligned} \tag{4.20}$$

We now estimate each of the term in (4.20). First, it is obvious that

$$\sum_{n=2}^{R-1} (A(w^{n+1}; u^{n+1}, \Pi^* u_h^{n+1}) - A(w^n; u^n, \Pi^* u_h^n)) = A(w^R; u^R, \Pi^* u_h^R) - A(w^2; u^2, \Pi^* u_h^2).$$

Second, it follows from Lemma 6 that

$$\left|\sum_{n=2}^{R-1} A(w^n; u^{n+1}, \Pi^* u_h^n) - A(w^n; u^n, \Pi^* u_h^n)\right| = \Delta t \left|\sum_{n=2}^{R-1} A(w^n; \partial_t u^{n+1}, \Pi^* u_h^n)\right|$$
$$\leqslant M\Delta t \sum_{n=2}^{R-1} (h^2 \|\partial_t u^{n+1}\|_{H^3} + \|\partial_t u^{n+1}\|_{H^1})\|u_h^n\|_{H^1}.$$

Third, from the argument in Lemma 6 we see that

$$\left|\sum_{n=2}^{R-1} A(w^{n+1}; u^{n+1}, \Pi^* u_h^n) - A(w^n; u^{n+1}, \Pi^* u_h^n)\right|$$
$$\leqslant M\Delta t \|\partial_t w^{n+1}\|_{L^\infty} \sum_{n=2}^{R-1} (h^2\|u^{n+1}\|_{H^3} + \|u^{n+1}\|_{H^1})\|u_h^n\|_{H^1}.$$

Finally, we combine the above three estimates to complete the proof. □

Next, we shall prove the error estimates of scheme (4.2)–(4.4). The analysis procedure is tedious but not complicated. Some common techniques, such as the Cauchy–Schwartz inequality, interpolation estimate, inverse estimate and induction hypothesis, are employed. We obtain the optimal $H^1$-norm error estimates of second–order for the approximate solutions.

**Theorem 1.** *Suppose that $\rho \geqslant \frac{2}{3} a^*$ and $h = O(\Delta t)$. Let $u_h$ be the solution of* (4.2)–(4.4). *Under Assumption* 1, *for sufficiently small h and $\Delta t$, there exists a constant $M > 0$, depending upon various norms of u, such that*

$$\sup_{0\leqslant n\leqslant N} \|u^n - u_h^n\|_{H^1} \leqslant M(\Delta t^2 + h^2). \tag{4.21}$$

**Proof.** Let $\xi = u_h - \Pi u$ and $\eta = u - \Pi u$. Obviously, $\xi^0 = 0$. Multiply (2.6) ($t = t_{n+1}$) by $\frac{2}{3}$, and then subtract the result to (4.2) to get the following error equation:

$$\begin{aligned}
&\left(\frac{\xi^{n+1} - \hat{\xi}^n}{\Delta t}, v\right) + \frac{2}{3} A(Eu_h^{n+1}; E\xi^{n+1}, v) + \rho A(\delta^2 \xi^{n+1}, v) + \rho^2 \Delta t B(\delta^2 \xi^{n+1}, v)\\
&= \frac{1}{3}\left(\frac{\hat{\xi}^n - \hat{\hat{\xi}}^{n-1}}{\Delta t}, v\right) + \left(\frac{\eta^{n+1} - \hat{\eta}^n}{\Delta t}, v\right) - \frac{1}{3}\left(\frac{\hat{\eta}^n - \hat{\hat{\eta}}^{n-1}}{\Delta t}, v\right)\\
&\quad + \left(\frac{2}{3}\psi(X, u^{n+1})\frac{\partial u^{n+1}}{\partial \tau} - \frac{u^{n+1} - \check{u}^n}{\Delta t} + \frac{1}{3}\frac{\check{u}^n - \check{\check{u}}^{n-1}}{\Delta t}, v\right)\\
&\quad + \frac{4}{3}\left(\frac{\hat{u}^n - \check{u}^n}{\Delta t}, v\right) - \frac{1}{3}\left(\frac{\hat{\hat{u}}^{n-1} - \check{\check{u}}^{n-1}}{\Delta t}, v\right)\\
&\quad + \tfrac{2}{3} A(Eu_h^{n+1}; \delta^2 u^{n+1}, v) + \tfrac{2}{3} A(Eu_h^{n+1}; E\eta^{n+1}, v)\\
&\quad + \tfrac{2}{3}(A(u^{n+1}; u^{n+1}, v) - A(Eu_h^{n+1}; u^{n+1}, v))\\
&\quad + \rho A(\delta^2 \Pi u^{n+1}, v) + \rho^2 \Delta t B(\delta^2 \Pi u^{n+1}, v)\\
&\quad + \tfrac{2}{3}(f^{n+1}(Eu_h^{n+1}) - f^{n+1}(u^{n+1}), v),
\end{aligned} \tag{4.22}$$

where $\check{u}^n = u^n(\check{X}) = u^n(X - b(X, u^{n+1})\Delta t)$ and $\check{\check{u}}^{n-1} = u^{n-1}(\check{\check{X}}) = u^{n-1}(X - 2b(X, u^{n+1})\Delta t)$.

Take $v=\Delta t\Pi^*\partial_t\xi^{n+1}$ as the test function in (4.22) and then sum the result for $1\leqslant n\leqslant R-1$, to estimate the terms on the right-hand side of (4.22). From [8] we have

$$\left\|\frac{\xi^n-\hat{\xi}^n}{\Delta t}\right\|_{L^2}\leqslant M\|\xi^n\|_{H^1}\quad\text{and}\quad\left\|\frac{\xi^{n-1}-\hat{\hat{\xi}}^{n-1}}{\Delta t}\right\|_{L^2}\leqslant M\|\xi^{n-1}\|_{H^1}.$$

Then by Lemmas 1, 2 , the Cauchy–Schwartz inequality and the Young inequality, we have

$$\begin{aligned}
&\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\xi^{n+1}-\hat{\xi}^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\\
&\quad=\frac{\Delta t}{3}\sum_{n=1}^{R-1}(\partial_t\xi^n,\Pi^*\partial_t\xi^{n+1})+\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\hat{\xi}^n-\xi^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)+\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\xi^{n-1}-\hat{\hat{\xi}}^{n-1}}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\\
&\quad\leqslant\frac{\Delta t}{3}\sum_{n=1}^{R-1}|||\partial_t\xi^n|||_0|||\partial_t\xi^{n+1}|||_0+M\Delta t\sum_{n=1}^{R-1}(\|\xi^n\|_{H^1}+\|\xi^{n-1}\|_{H^1})\|\partial_t\xi^{n+1}\|_{0,h}\\
&\quad\leqslant\frac{\Delta t}{6}\sum_{n=1}^{R-1}(|||\partial_t\xi^n|||_0^2+|||\partial_t\xi^{n+1}|||_0^2)+M\Delta t\sum_{n=1}^{R-1}(\|\xi^n\|_{H^1}^2+\|\xi^{n-1}\|_{H^1}^2)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2\\
&\quad\leqslant\frac{\Delta t}{3}\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2+\frac{\Delta t}{6}|||\partial_t\xi^1|||_0^2+2M\Delta t\sum_{n=1}^{R-1}(\|\xi^{n+1}\|_{H^1}^2+\|\xi^1\|_{H^1}^2)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2.
\end{aligned}\tag{4.23}$$

Similarly,

$$\begin{aligned}
&\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\eta^{n+1}-\hat{\eta}^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\\
&\quad=\frac{\Delta t}{3}\sum_{n=1}^{R-1}(\partial_t\eta^{n+1},\Pi^*\partial_t\xi^{n+1})+\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\eta^n-\hat{\eta}^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\\
&\quad\leqslant M\Delta t\sum_{n=1}^{R-1}\left(\|\partial_t\eta^{n+1}\|_{L^2}^2+\left\|\frac{\eta^n-\hat{\eta}^n}{\Delta t}\right\|_{L^2}^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2\\
&\quad\leqslant M\{\|u\|_{H^1(H^3},\|u\|_{L^\infty(H^3}\}h^4+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2,
\end{aligned}\tag{4.24}$$

and

$$\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\hat{\eta}^n-\hat{\hat{\eta}}^{n-1}}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\leqslant M\{\|u\|_{H^1(H^3(\Omega)},\|u\|_{L^\infty(H^3(\Omega)}\}h^4+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2.\tag{4.25}$$

Integrating along characteristic direction as in [20] gives

$$
\begin{aligned}
&\left\| \frac{2}{3}\psi(X,u^{n+1})\frac{\partial u^{n+1}}{\partial\tau} - \frac{u^{n+1}-\check{u}^n}{\Delta t} + \frac{1}{3}\frac{\check{u}^n-\check{\check{u}}^{n-1}}{\Delta t}\right\|_{L^2} \\
&\quad\leqslant \frac{M}{\Delta t}\left\| \int_{(\check{X},t_n)}^{(X,t_{n+1})} (|X(\tau)-\check{X}|^2+(t(\tau)-t_n)^2)\frac{\partial^3 u}{\partial\tau^3}\,\mathrm{d}\tau \right.\\
&\qquad\left. + \int_{(\check{\check{X}},t_{n-1})}^{(X,t_{n+1})} (|X(\tau)-\check{\check{X}}|^2+(t(\tau)-t_{n-1})^2)\frac{\partial^3 u}{\partial\tau^3}\,\mathrm{d}\tau \right\|_{L^2}
\end{aligned}
$$

Then, it follows from the Cauchy-Schwartz inequality and the Young inequality that

$$
\begin{aligned}
&\Delta t\sum_{n=1}^{R-1}\left( \frac{2}{3}\psi(X,u^{n+1})\frac{\partial u^{n+1}}{\partial\tau} - \frac{u^{n+1}-\check{u}^n}{\Delta t} + \frac{1}{3}\frac{\check{u}^n-\check{\check{u}}^{n-1}}{\Delta t}, \Pi^*\partial_t\xi^{n+1}\right) \\
&\quad\leqslant M\left\|\frac{\partial^3 u}{\partial\tau^3}\right\|^2_{L^2(L^2)}\Delta t^4 + \varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|^2_{L^2}.
\end{aligned}
\tag{4.26}
$$

From Assumption 1, we know that $b(X,v)$ is only $\varepsilon_0$-continuous. So in order to estimate the terms related to $b$, we shall make the following induction hypothesis:

$$
\sup_{0\leqslant n\leqslant R-1}\|\xi^n\|_{L^\infty}\leqslant h^{3/2}, \quad (h,\Delta t)\longrightarrow 0. \tag{4.27}
$$

From (4.27), we see that when $u\in L^\infty(0,T;W^{1,\infty}(\Omega))\cap W^{2,\infty}(0,T;L^\infty(\Omega))$, for sufficiently small $h$ and $\Delta t$,

$$
\begin{aligned}
\sup_{0\leqslant n\leqslant R-1}\|Eu_h^{n+1}-u^{n+1}\|_{L^\infty} &\leqslant \sup_{0\leqslant n\leqslant R-1}(\|E\xi^{n+1}\|_{L^\infty}+\|E\eta^{n+1}\|_{L^\infty}+\|\delta u^{n+1}\|_{L^\infty}) \\
&\leqslant \sup_{0\leqslant n\leqslant R-1} 3\|\xi^n\|_{L^\infty} + M_u(h+\Delta t^2)\leqslant\varepsilon_0,
\end{aligned}
\tag{4.28}
$$

where $M_u>0$ is a fixed constant only depending on the norms of $u$. Then (4.28) with the definition of $\hat{u}$ and $\check{u}$ gives

$$
\begin{aligned}
\frac{\hat{u}^n-\check{u}^n}{\Delta t} &\leqslant M\{u_{L^\infty(W^{1,\infty})}\}\frac{|\hat{X}-\check{X}|}{\Delta t} = M\{u_{L^\infty(W^{1,\infty})}\}|b(X,Eu_h^{n+1})-b(X,u^{n+1})| \\
&\leqslant M\{u_{L^\infty(W^{1,\infty})},L\}|Eu_h^{n+1}-u^{n+1}|.
\end{aligned}
$$

Thus, we use this estimate, the interpolation estimate (2.6) and the Young inequality to get

$$
\begin{aligned}
&\frac{4\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\hat{u}^n-\check{u}^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right) \\
&\quad\leqslant M\{u_{L^\infty(W^{1,\infty})},L\}\Delta t\sum_{n=1}^{R-1}\|Eu_h^{n+1}-u^{n+1}\|_{L^2}\|\partial_t\xi^{n+1}\|_{L^2} \\
&\quad\leqslant M\{u_{L^\infty(W^{1,\infty})}\}\Delta t\sum_{n=1}^{R-1}(\|E\xi^{n+1}\|_{L^2}+\|E\eta^{n+1}\|_{L^2}+\|\delta^2u^{n+1}\|_{L^2})\|\partial_t\xi^{n+1}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant M\{u_{L^\infty(W^{1,\infty})}, u_{L^\infty(H^3)}, u_{H^2(L^2)}, L\}\left(h^6+\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|E\xi^{n+1}\|_{L^2}^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2\\
&\leqslant M\left(h^6+\Delta t^4+\Delta t\|\xi^1\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2.
\end{aligned}
\tag{4.29}
$$

It follows from a same way that

$$
\begin{aligned}
-\frac{\Delta t}{3}\sum_{n=1}^{R-1}\left(\frac{\hat{\hat{u}}^n-\check{\check{u}}^n}{\Delta t}, \Pi^*\partial_t\xi^{n+1}\right)\leqslant{}& M\{u_{L^\infty(W^{1,\infty})}, u_{L^\infty(H^3)}, u_{W^2(L^2)}, L\}\\
&\times\left(h^6+\Delta t^4+\Delta t\|\xi^1\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2\right)\\
&+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2.
\end{aligned}
\tag{4.30}
$$

We now make the second hypothesis that

$$
\sup_{0\leqslant n\leqslant R-1}\|\partial_t u_h^n\|_{L^\infty}\leqslant M^*,\quad (h,\Delta t)\longrightarrow 0,
\tag{4.31}
$$

where $M^*$ is fixed constant to be given in later. Then from the triangle inequality, we know the following estimate is true:

$$
\sup_{0\leqslant n\leqslant R-1}\|E\partial_t u_h^{n+1}\|_{L^\infty}\leqslant\sup_{0\leqslant n\leqslant R-1}3\|\partial_t u_h^n\|_{L^\infty}\leqslant 3M^*,\quad (h,\Delta t)\longrightarrow 0.
\tag{4.32}
$$

Therefore, using (4.32), Lemma 13 and the Young inequality, we have

$$
\begin{aligned}
&\frac{2\Delta t}{3}\sum_{n=1}^{R-1}A(Eu_h^{n+1};\delta^2u^{n+1},\Pi^*\partial_t\xi^{n+1})\\
&\leqslant A(Eu_h^R;\delta^2u^R,\Pi^*\xi^R)-A(Eu_h^2;\delta^2u^2,\Pi^*\xi^1)+M\Delta t\sum_{n=2}^{R-1}(h^2[\|\delta^2u^{n+1}\|_{H^3}\\
&\quad+\|\partial_t(\delta^2u^{n+1})\|_{H^3}]+\|\partial_t(\delta^2u^{n+1})\|_{H^1}+\|\delta^2u^{n+1}\|_{H^1})\|\xi^n\|_{H^1}\\
&\leqslant A(Eu_h^R;\delta^2u^R,\Pi^*\xi^R)-A(Eu_h^2;\delta^2u^2,\Pi^*\xi^1)\\
&\quad+M\{\|u\|_{H^1(H^3)},\|u\|_{H^3(H^1)}\}\left(h^4+\Delta t^4+\Delta t\sum_{n=2}^{R-1}\|\xi^n\|_{H^1}^2\right).
\end{aligned}
$$

For the first two terms in the right-hand side of the above inequality, using Lemma 6 and the Young inequality gives the following estimates:

$$
\begin{aligned}
&|A(Eu_h^R;\delta^2u^R,\Pi^*\xi^R)-A(Eu_h^2;\delta^2u^2,\Pi^*\xi^1)|\\
&\quad\leqslant M(h^2\|\delta^2u^R\|_{H^3}+h^2\|\delta^2u^2\|_{H^3}+\|\delta^2u^R\|_{H^1}+\|\delta^2u^2\|_{H^1})(\|\xi^R\|_{H^1}+\|\xi^1\|_{H^1})\\
&\quad\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{W^{2,\infty}(H^1)}\}(h^2+\Delta t^2)(\|\xi^R\|_{H^1}+\|\xi^1\|_{H^1})\\
&\quad\leqslant M(h^4+\Delta t^4+\|\xi^1\|_{H^1}^2)+\varepsilon\|\xi^R\|_{H^1}^2.
\end{aligned}
$$

Thus, combining the above two estimates, we have

$$\frac{2\Delta t}{3}\sum_{n=1}^{R-1} A(Eu_h^{n+1};\delta^2 u^{n+1},\Pi^*\partial_t\xi^{n+1})\leqslant M\left(h^4++\Delta t^4+\Delta t\sum_{n=2}^{R-1}\|\xi^n\|_{H^1}^2+\|\xi^1\|_{H^1}^2\right)+\varepsilon\|\xi^R\|_{H^1}^2. \quad (4.33)$$

Using the interpolation estimate (2.6) and proceeding similarly as in (4.33) yields

$$\begin{aligned}&\frac{2\Delta t}{3}\sum_{n=1}^{R-1} A(Eu_h^{n+1};E\eta^{n+1},\Pi^*\partial_t\xi^{n+1})\\ &\quad\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{H^1(H^3)}\}\left(h^4+\Delta t\sum_{n=2}^{R-1}\|\xi^n\|_{H^1}^2+\|\xi^1\|_{H^1}^2\right)+\varepsilon\|\xi^R\|_{H^1}^2. \quad (4.34)\end{aligned}$$

We note that

$$\begin{aligned}&\frac{2\Delta t}{3}\sum_{n=1}^{R-1}(A(u^{n+1};u^{n+1},\Pi^*\partial_t\xi^{n+1})-A(Eu_h^{n+1};u^{n+1},\Pi^*\partial_t\xi^{n+1}))\\ &\quad=\frac{2}{3}([A(u^R;u^R,\Pi^*\xi^R)-A(Eu_h^R;u^R,\Pi^*\xi^R)]+[A(u^2;u^2,\Pi^*\xi^1)-A(Eu_h^2;u^2,\Pi^*\xi^1)]\\ &\qquad-\sum_{n=2}^{R-1}[A(u^{n+1};\delta u^{n+1},\Pi^*\xi^n)-A(Eu_h^{n+1};\delta u^{n+1},\Pi^*\xi^n)]\\ &\qquad-\sum_{n=2}^{R-1}[(A(u^{n+1};u^n,\Pi^*\xi^n)-A(Eu_h^{n+1};u^n,\Pi^*\xi^n)).\end{aligned}$$

First, for the first three terms on the above equation, it follows from Lemma 7 that

$$\begin{aligned}A(u^R;u^R,\Pi^*\xi^R)-A(Eu_h^R;u^R,\Pi^*\xi^R)&\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{L^\infty(W^{1,\infty})}\}(h^3+\|u^R-Eu_h^R\|_{L^2})\|\xi^R\|_{H^1}\\ &\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{L^\infty(W^{1,\infty})}\}(h^3+\Delta t^2+\|E\xi^R\|_{L^2})\|\xi^R\|_{H^1}.\end{aligned}$$

$$\begin{aligned}A(u^2;u^2,\Pi^*\xi^1)-A(Eu_h^2;u^2,\Pi^*\xi^1)&\leqslant M(h^3+\|u^2-Eu_h^2\|_{L^2})\|\xi^1\|_{H^1}\\ &\leqslant M(h^3+\Delta t^2+\|\xi^1\|_{L^2})\|\xi^1\|_{H^1}.\end{aligned}$$

$$\begin{aligned}&\left|\sum_{n=2}^{R-1}[A(u^{n+1};\delta u^{n+1},\Pi^*\xi^n)-A(Eu_h^{n+1};\delta u^{n+1},\Pi^*\xi^n)]\right|\\ &\quad\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{W^{1,\infty}(W^{1,\infty})}\Delta t\sum_{n=2}^{R-1}(h^3+\Delta t^2+\|\xi^n\|_{L^2}+\|\xi^{n-1}\|_{L^2})\|\xi^n\|_{H^1}.\end{aligned}$$

Second, we note that

$$\delta[a(u^{n+1})-a(Eu_h^{n+1})]=\frac{\partial\tilde a}{\partial v}\delta u^{n+1}-\frac{\partial\bar a}{\partial v}\delta Eu_h^{n+1}=\frac{\partial\bar a}{\partial v}\delta(u^{n+1}-Eu_h^{n+1})+\left(\frac{\partial\tilde a}{\partial v}-\frac{\partial\bar a}{\partial v}\right)\delta u^{n+1},$$

where $\tilde{a}=a(\theta_n u^n+(1-\theta_n)u^{n+1})$, $\bar{a}=a(\theta_n Eu_h^n+(1-\theta_n)Eu_h^{n+1})$ and $0<\theta_n<1$. Thus, it follows from the Assumption 1 and the argument in Lemma 7 that

$$\begin{aligned}
&|A(u^{n+1};u^n,\Pi^*\xi^n)-A(Eu_h^{n+1};u^n,\Pi^*\xi^n)-(A(u^n;u^n,\Pi^*\xi^n)-A(Eu_h^n;u^n,\Pi^*\xi^n))|\\
&\quad\leqslant Ma^*\|\delta u^{n+1}\|_{W^{1,\infty}}(h^3\|\delta u^{n+1}\|_{L^2}+\|\delta(u^{n+1}-Eu_h^{n+1})\|_{L^2})\|\xi^n\|_{H^1}\\
&\qquad+Ma^*\|\delta u^{n+1}\|^2_{W^{1,\infty}}(h^3\|\theta_n u^n+(1-\theta_n)u^{n+1}\|_{L^2}\\
&\qquad+\|\theta_n(u^n-Eu_h^n)+(1-\theta_n)(u^{n+1}-Eu_h^{n+1})\|_{L^2})\|\xi^n\|_{H^1}\\
&\quad\leqslant M\{\|u\|_{W^{1,\infty}(W^{1,\infty})},\|u\|_{L^\infty(H^3)}\}\Delta t(h^3+\|E\xi^n\|_{L^2}+\|E\xi^{n+1}\|_{L^2})\|\xi^n\|_{H^1}.
\end{aligned}$$

Finally, we combine the above four estimates and use the Young inequality to see that

$$\begin{aligned}
&\frac{2\Delta t}{3}\sum_{n=1}^{R-1}(A(u^{n+1};u^{n+1},\Pi^*\partial_t\xi^{n+1})-A(Eu_h^{n+1};u^{n+1},\Pi^*\partial_t\xi^{n+1}))\\
&\quad\leqslant M\left(h^6+\Delta t^4+\|E\xi^R\|^2_{L^2}+\|\xi^1\|^2_{H^1}+\Delta t\|\xi^1\|^2_{H^1}+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|^2_{H^1}\right)+\varepsilon\|\xi^R\|^2_{H^1}.
\end{aligned}\tag{4.35}$$

It is obvious that $\delta^2(\Pi u^{n+1})=\delta^2u^{n+1}+\delta^2\eta^{n+1}$. Then proceeding similarly as in (4.33), we have

$$\begin{aligned}
&\rho\Delta t\sum_{n=1}^{R-1}A(\delta^2\Pi u^{n+1},\Pi^*\partial_t\xi^{n+1})\\
&\quad=\rho\Delta t\sum_{n=1}^{R-1}A(\delta^2u^{n+1},\Pi^*\partial_t\xi^{n+1})+\rho\Delta t\sum_{n=1}^{R-1}A(\delta^2\eta^{n+1},\Pi^*\partial_t\xi^{n+1})\\
&\quad\leqslant M\left(h^4+\Delta t^4+\|\xi^1\|^2_{H^1}+\Delta t\sum_{n=2}^{R-1}\|\xi^n\|^2_{H^1}\right)+\varepsilon\|\xi^R\|^2_{H^1}.
\end{aligned}\tag{4.36}$$

According to (4.6) and using integral by parts and the inverse estimate [17], we see that

$$\begin{aligned}
&\rho^2\Delta t^2\sum_{n=1}^{R-1}B(\delta^2\Pi u^{n+1},\Pi^*\partial_t\xi^{n+1})=-\rho^2\Delta t\sum_{n=1}^{R-1}\left(\frac{\partial^2(\delta^2\Pi u^{n+1})}{\partial x\partial y},\frac{\partial^2(\delta\xi^{n+1})}{\partial x\partial y}\right)\\
&\quad=\rho^2\Delta t\sum_{n=1}^{R-1}\left[\left(\frac{\partial^3(\delta^2u^{n+1})}{\partial^2x\partial y},\frac{\partial(\delta\xi^{n+1})}{\partial y}\right)+\left(\frac{\partial^2(\delta^2\eta^{n+1})}{\partial x\partial y},\frac{\partial^2(\delta\xi^{n+1})}{\partial x\partial y}\right)\right]\\
&\quad\leqslant\rho^2\Delta t\sum_{n=1}^{R-1}(\|\delta^2u^{n+1}\|_{H^3}\|\delta\xi^{n+1}\|_{H^1}+\|\delta^2\eta^{n+1}\|_{H^2}\|\delta\xi^{n+1}\|_{H^2})\\
&\quad\leqslant M\{\rho,\|u\|_{H^2(H^3)}\}\left(\Delta t\sum_{n=1}^{R-1}(\|\delta^2u^{n+1}\|^2_{H^3}+\|\delta\xi^{n+1}\|^2_{H^1})+\Delta t\sum_{n=1}^{R-1}(h^{-2}\|\delta^2\eta^{n+1}\|^2_{H^2}+\|\delta\xi^{n+1}\|^2_{H^1})\right)\\
&\quad\leqslant M\{\rho,\|u\|_{H^2(H^3)}\}\left(\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|\delta\xi^{n+1}\|^2_{H^1}\right).
\end{aligned}\tag{4.37}$$

At last, from the $\varepsilon_0$-Lipschitz continuity of $f$ and (4.28), for sufficiently small $h$ and $\Delta t$ small enough, we use the triangle inequality and the Young inequality to yield

$$\begin{aligned}
&\frac{2\Delta t}{3}\sum_{n=1}^{R-1}(f^{n+1}(Eu_h^{n+1})-f^{n+1}(u^{n+1}),\Pi^*\partial t\,\xi^{n+1})\\
&\leqslant M\Delta t\sum_{n=1}^{R-1}\|Eu_h^{n+1}-u^{n+1}\|_{L^2}^2+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial t\,\xi^{n+1}\|_{L^2}^2\\
&\leqslant M\{\|u\|_{L^\infty(H^3)},\|u\|_{H^2(L^2)}\}\left(\Delta t^4+h^6+\Delta t\sum_{n=1}^{R-1}\|E\xi^{n+1}\|_{L^2}^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial t\,\xi^{n+1}\|_{L^2}^2\\
&\leqslant M\left(\Delta t^4+h^6+\Delta t\|\xi^1\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial t\,\xi^{n+1}\|_{L^2}^2.
\end{aligned}\tag{4.38}$$

Now we begin to estimate the terms on the left-hand side of (4.22). First similar as in (4.23), we have

$$\begin{aligned}
&\sum_{n=1}^{R-1}(\xi^{n+1}-\hat{\xi}^n,\Pi^*\partial_t\xi^{n+1})\\
&=\Delta t\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2+\Delta t\sum_{n=1}^{R-1}\left(\frac{\xi^n-\hat{\xi}^n}{\Delta t},\Pi^*\partial_t\xi^{n+1}\right)\\
&\geqslant\Delta t\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2-M\Delta t\sum_{n=1}^{R-1}\|\xi^n\|_{H^1}\|\partial_t\xi^{n+1}\|_{L^2}\\
&\geqslant\Delta t\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2-M\Delta t\sum_{n=1}^{R-1}\|\xi^n\|_{H^1}^2-\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}.
\end{aligned}\tag{4.39}$$

Next, for the second two terms on the left-hand side of (4.22), we note that

$$\begin{aligned}
&\frac{2\Delta t}{3}\sum_{n=1}^{R-1}A(Eu_h^{n+1};E\xi^{n+1},\Pi^*\partial_t\xi^{n+1})+\rho\Delta t\sum_{n=1}^{R-1}A(\delta^2\xi^{n+1},\Pi^*\partial_t\xi^{n+1})\\
&\quad=\frac{2}{3}\sum_{n=1}^{R-1}A(Eu_h^{n+1};\xi^{n+1},\Pi^*\delta\xi^{n+1})+\sum_{n=1}^{R-1}\left(\rho A(\delta^2\xi^{n+1},\Pi^*\delta\xi^{n+1})-\frac{2}{3}A(Eu_h^{n+1};\delta^2\xi^{n+1},\Pi^*\delta\xi^{n+1})\right),
\end{aligned}$$

where from (4.28), (4.32) and the triangle inequality, we see that $Eu_h^{n+1}\in L^\infty(\Omega)$ and $E\partial_t u_h^{n+1}\in L^\infty(\Omega)$ if $u^{n+1}\in L^\infty(\Omega)$. Thus, we use Lemma 11 to get

$$\frac{2}{3}\sum_{n=1}^{R-1}A(Eu_h^{n+1};\xi^{n+1},\Pi^*\delta\xi^{n+1})\geqslant\frac{1}{3}A(Eu_h^R;\xi^R,\Pi^*\xi^R)-M\left(\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{H^1}^2+\|\xi^1\|_{H^1}^2\right),$$

and use Lemma 12 to get

$$\begin{aligned}
&\sum_{n=1}^{R-1}(\rho A(\delta^2\xi^{n+1},\Pi^*\delta\xi^{n+1})-\frac{2}{3}A(Eu_h^{n+1};\delta^2\xi^{n+1},\Pi^*\delta\xi^{n+1}))\\
&\quad\geqslant-M\left(\Delta t\sum_{n=1}^{R-1}\|\delta\xi^{n+1}\|_{H^1}^2+\|\delta\xi^1\|_{H^1}^2\right)\geqslant-M\left(\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{H^1}^2+\|\xi^1\|_{H^1}^2\right).
\end{aligned}$$

Hence, combining the above three estimates and using Lemma 8, for sufficiently small $h$, we obtain the following inequality:

$$\begin{aligned}&\frac{2\Delta t}{3}\sum_{n=1}^{R-1} A(Eu_h^{n+1}; E\xi^{n+1}, \Pi^*\partial_t\xi^{n+1}) + \rho\Delta t\sum_{n=1}^{R-1} A(\delta^2\xi^{n+1}, \Pi^*\partial_t\xi^{n+1})\\ &\quad \geqslant M_0\|\xi^R\|_{H^1}^2 - M\left(\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{H^1}^2 + \|\xi^1\|_{H^1}^2\right),\end{aligned} \tag{4.40}$$

where $M_0 > 0$ is a fixed constant.

At last, using summation by parts in temporal direction, it is easy to see that

$$\rho^2\Delta t^2\sum_{n=1}^{R-1} B(\delta^2\xi^{n+1}, \Pi^*\partial_t\xi^{n+1}))\geqslant\frac{\rho^2\Delta t}{2}(B(\delta\xi^R, \Pi^*\delta\xi^R) - B(\delta\xi^1, \Pi^*\delta\xi^1)). \tag{4.41}$$

Finally, we combine (4.23)–(4.41) and choose ε small enough to obtain

$$\begin{aligned}&\|\xi^R\|_{H^1}^2 + \Delta t\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2 + \Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2 + \Delta t B(\delta\xi^R, \Pi^*\delta\xi^R)\\ &\quad\leqslant M\left(\Delta t^4 + h^4 + \Delta t|||\partial_t\xi^1|||_0^2 + \|\xi^1\|_{H^1}^2 + \Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{H^1}^2 + \|E\xi^R\|_{L^2}^2 + \Delta t B(\delta\xi^1, \Pi^*\delta\xi^1)\right).\end{aligned}$$

From [9] we see that $\|E\xi^R\|_{L^2}^2 \leqslant \|E\xi^1\|_{L^2}^2 + \varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t E\xi^{n+1}\|_{L^2}^2 + M\Delta t\sum_{n=1}^{R-1}\|E\xi^{n+1}\|_{L^2}^2$. Thus, using the triangle inequality yields

$$\|E\xi^R\|_{L^2}^2\leqslant 4\|\xi^1\|_{L^2}^2 + 9\varepsilon\Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2 + \|\xi^1\|_{L^2}^2 + M\Delta t(\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2 + \|\xi^1\|_{L^2}^2).$$

Then from the above two inequality and for sufficiently small ε, we apply the Gronwall lemma to have

$$\begin{aligned}&\|\xi^R\|_{H^1}^2 + \Delta t\sum_{n=1}^{R-1}|||\partial_t\xi^{n+1}|||_0^2 + \Delta t\sum_{n=1}^{R-1}\|\partial_t\xi^{n+1}\|_{L^2}^2 + \Delta t B(\delta\xi^R, \Pi^*\delta\xi^R)\\ &\quad\leqslant M(\Delta t^4 + h^4 + \Delta t|||\partial_t\xi^1|||_0^2 + \|\xi^1\|_{H^1}^2 + \Delta t B(\delta\xi^1, \Pi^*\delta\xi^1)).\end{aligned} \tag{4.42}$$

For (4.4) a similar argument as in this theorem gives

$$\|\xi^1\|_{H^1}^2 + \Delta t|||\partial_t\xi^1|||_0^2 + \Delta t B(\delta\xi^1, \Pi^*\delta\xi^1)\leqslant M(\Delta t^4 + h^4).$$

Combining this approximation with (4.42), we have

$$\|\xi^R\|_{H^1}^2\leqslant M(\Delta t^4 + h^4). \tag{4.43}$$

Since $\xi^0 = 0$ and $\partial_t u_h^0 = 0$, it is obvious that (4.27) and (4.31) hold for $n = 0$. Now we shall verify the hypothesis (4.27) and (4.31) hold for $n = R$. In fact, when the discretization parameters satisfy the relation $h = \mathrm{O}(\Delta t)$, Using (4.43) and the inverse estimate, we see that

$$\|\xi^R\|_{L^\infty}\leqslant M|\ln h|^{1/2}\|\xi^R\|_{H^1}\leqslant M(|\ln h|h)^{1/2}h^{\frac{3}{2}}\leqslant h^{3/2}, \quad h\longrightarrow 0.$$

Next, the above estimate with the induction hypothesis (4.27) gives

$$\|\partial_t \xi^R\|_{L^\infty} = \Delta t^{-1}\|\xi^R - \xi^{R-1}\|_{L^\infty} \leqslant \Delta t^{-1}\|\xi^R\|_{L^\infty} + \Delta t^{-1}\|\xi^{R-1}\|_{L^\infty} \leqslant 2\Delta t^{-1} h^{\frac{3}{2}} \leqslant 1, \quad (h, \Delta t) \longrightarrow 0.$$

So let $M^* = \|\partial_t \Pi u\|_{L^\infty(L^\infty)} + 1$ in (4.31). Then

$$\|\partial_t u_h^R\|_{L^\infty} \leqslant \|\partial_t \Pi u^n\|_{L^\infty} + \|\partial_t \xi^n\|_{L^\infty} \leqslant M^*.$$

Hence when $h$ and $\Delta t$ are small enough and $h = \mathrm{O}(\Delta t)$, (4.27) and (4.31) hold for all $n$. Note that in (4.43) $R$ is a generic integer not larger than $N$. Thus, combining (4.43) with the interpolation estimate (2.6), we complete the proof of the theorem. □

## 5. Numerical experiment

In this section, we consider a convection diffusion problem. We discretize it by use of the FVE method proposed in this paper. We compare the numerical results produced by the single step scheme ($q = 1$) and the multistep scheme ($q = 2$). The numerical results show that under the same condition, the results produced by the multistep scheme are more accurate than those produced by the single step one. Consider the following problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + b \cdot \nabla u = f(X, t), \quad (X, t) \in \Omega \times (0, 1], \\ &u = 0, \quad (X, t) \in \partial\Omega \times (0, 1], \\ &u(X, 0) = x(1-x)y(1-y), \quad X \in \Omega, \end{aligned} \tag{5.1}$$

where $\Omega = (0, 1)^2$ and $b = (\sin \pi x \sin \pi y, \sin \pi x \sin \pi y)$. The exact solution $u = \mathrm{e}^{-t} x(1-x)y(1-y)$. We choose a constant spatial step $h$. Then some numerical results ($t = \frac{1}{2}$) are listed in Tables 1–3 for various $h$ and $\Delta t$.

Table 1
$h = 1/8$, $\Delta t = 1/10$

| Nodes | (1/8, 1/8) | (1/8, 1/4) | (1/8, 3/8) | (1/8, 1/2) | (1/4, 1/4) | (1/4, 3/8) | (1/4, 1/2) | (1/2, 1/2) |
|---|---|---|---|---|---|---|---|---|
| $u$ | 0.007256 | 0.012439 | 0.015548 | 0.016585 | 0.021323 | 0.026654 | 0.028431 | 0.037908 |
| $u_h\|_{q=1}$ | 0.006826 | 0.011645 | 0.014514 | 0.015468 | 0.019879 | 0.024783 | 0.026415 | 0.035106 |
| $u_h\|_{q=2}$ | 0.007372 | 0.012558 | 0.015639 | 0.016665 | 0.021409 | 0.026673 | 0.028429 | 0.037773 |

Table 2
$h = 1/16$, $\Delta t = 1/20$

| Nodes | (1/8, 1/8) | (1/8, 1/4) | (1/8, 3/8) | (1/8, 1/2) | (1/4, 1/4) | (1/4, 3/8) | (1/4, 1/2) | (1/2, 1/2) |
|---|---|---|---|---|---|---|---|---|
| $u$ | 0.007256 | 0.012439 | 0.015548 | 0.016585 | 0.021323 | 0.026654 | 0.028431 | 0.037908 |
| $u_h\|_{q=1}$ | 0.007010 | 0.011995 | 0.014975 | 0.015968 | 0.020525 | 0.025626 | 0.027325 | 0.036377 |
| $u_h\|_{q=2}$ | 0.007237 | 0.012735 | 0.015445 | 0.016468 | 0.021168 | 0.026424 | 0.028175 | 0.037506 |

Table 3
$h = 1/32$, $\Delta t = 1/40$

| Nodes | (1/8, 1/8) | (1/8, 1/4) | (1/8, 3/8) | (1/8, 1/2) | (1/4, 1/4) | (1/4, 3/8) | (1/4, 1/2) | (1/2, 1/2) |
|---|---|---|---|---|---|---|---|---|
| $u$ | 0.007256 | 0.012439 | 0.015548 | 0.016585 | 0.021323 | 0.026654 | 0.028431 | 0.037908 |
| $u_h\|_{q=1}$ | 0.007125 | 0.012204 | 0.015247 | 0.016261 | 0.020904 | 0.026116 | 0.027852 | 0.037108 |
| $u_h\|_{q=2}$ | 0.007227 | 0.012377 | 0.015461 | 0.016489 | 0.021198 | 0.026481 | 0.028241 | 0.037626 |

When $h=\frac{1}{8}$, $\Delta t=\frac{1}{10}$, for the single step scheme, the maximal absolute error is 0.003796 and the maximal relative error is 7.42%, while for the multistep scheme the maximal absolute error is 0.000214 and the maximal relative error is 3.59%.

When $h=\frac{1}{16}$, $\Delta t=\frac{1}{20}$, for the single step scheme, the maximal absolute error is 0.002069 and the maximal relative error is 4.04%, while for the multistep scheme the maximal absolute error is 0.000490 and the maximal relative error is 1.06%.

When $h=\frac{1}{32}$, $\Delta t=\frac{1}{40}$, for the single step scheme, the maximal absolute error is 0.001073 and the maximal relative error is 2.11%, while for the multistep scheme the maximal absolute error is 0.000364 and the maximal relative error is 0.75%.

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